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Quantum correlations and the extended phase space

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Abstract
An extended phase space *x*–*p*–*X*–*P* is introduced for one-mode systems, where *x*–*p* is the position–momentum plane and *X*–*P* is the quantum correlations plane. The Wigner functions $W(x, p)$ and the Weyl functions $\tilde{W}(X, P)$ for arbitrary operators are studied in this space. Creation and annihilation operators for Wigner and Weyl functions are introduced and are used to perform displacements and squeezing in the *x*–*p*–*X*–*P* extended phase space. The formalism can be used for the construction of quantum states with desirable correlation properties.

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1. Introduction

The Wigner function $W(x, p)$ [1–3] describes a particle in phase space in a way consistent with the quantum mechanics. The Weyl function $\tilde{W}(X, P)$, where X and P are position and momentum *increments*, describes its correlation properties. In view of the central role that the quantum correlations play in quantum communications and quantum computing, it is important to develop this formalism further.

Recent work [4] has introduced the concept of extended phase space *x*–*p*–*X*–*P* (position–momentum–position increment–momentum increment) in order to provide a description that makes explicit not only the position and momentum of the particle but also its correlation properties. Quantum correlations are an important aspect of quantum mechanics and although they are best exemplified in Schrödinger cat states, they are present in every quantum state. The extended phase space formalism aims to describe quantum correlations in the 'correlation plane' *X*–*P* and investigate their relationship with the position and momentum in the *x*–*p* plane. Reference [4] has introduced novel quantities which describe the uncertainties in position and momentum $(\delta x, \delta p)$ and also the quantum correlations in position and momentum $(\delta X, \delta P)$. It proved novel uncertainty relations which elucidate the deep connection between uncertainties in position and momentum, and quantum correlations in position and momentum. Similar uncertainty relations have also been discussed in [5].

In this paper we extend further this work. In section 2 we introduce Wigner and Weyl functions for arbitrary operators which are not necessarily density matrices. Non-diagonal parts of a density matrix can be viewed as general non-Hermitian operators and the study of such operators in this paper enables us to manipulate individually the various components of a quantum state. We also introduce an orthonormal basis in terms of the Laguerre polynomials. Since the Wigner and Weyl functions are related through the Fourier transform we are able to construct a formalism similar to the harmonic oscillator formalism for wavefunctions. In section 3 we study various quantities that quantify the quantum correlations and discuss their relative merits.

In section 4 we introduce creation and annihilation operators for the Wigner and Weyl functions. The Wigner *x–p* and the Weyl *X–P* representations are similar to the position and momentum representations in the harmonic oscillator formalism. In section 5 we use them to introduce displacement operators in the *x–p–X–P* space, for the Wigner and Weyl functions. These displacement operators depend on four variables and displace not only in the position–momentum *x–p* space, but also in the quantum correlation plane *X–P*. In section 6 we study the Gaussian Wigner and Weyl functions and their displacements.

In section 7 we study squeezing of Wigner and Weyl functions. We stress that this is much more general than squeezing in the *x–p* phase space. It is squeezing in the *x–p–X–P* extended phase space and its general case involves ten generators. In this paper we only study a special case that demonstrates its use for the design of quantum states with desirable correlation properties.

In section 8 we use this formalism to study the time evolution of Wigner and Weyl functions. We conclude with a discussion of our results in section 9.

2. Wigner and Weyl functions

We consider the displacement operator $D(A)$,

$$D(A) = \exp(Aa^\dagger - A^*a) \tag{1}$$

where $a^\dagger$ and a are the usual creation and annihilation operators. $D(A)$ is a unitary operator, which can also be expressed in terms of the position $\hat{x}$ and momentum $\hat{p}$ operators as

$$D\left(A = \frac{x + \mathrm{i}p}{\sqrt{2}}\right) \equiv D(x, p) = \exp(\mathrm{i}p\hat{x} - \mathrm{i}x\hat{p}). \tag{2}$$

Coherent states are defined as $|\alpha_1, \alpha_2\rangle = D(\alpha_1, \alpha_2)|0\rangle$. For later purposes we point out that the matrix elements of $D(A)$ in terms of Laguerre polynomials [1] are given by

$$\langle M|D(A)|N\rangle = \left(\frac{N!}{M!}\right)^{1/2} A^{M-N} \exp\left(\frac{-|A|^2}{2}\right) L_N^{M-N}(|A|^2) \tag{3}$$

where L_N^{M-N} are Laguerre polynomials and $|N\rangle$ are number eigenstates with $N = 0, 1, 2, \ldots$. We also consider the parity operator

$$U_0 = \exp(\mathrm{i}\pi a^\dagger a) = \sum_N (-1)^N |N\rangle\langle N| \tag{4}$$

and the displaced parity operator

$$U(x, p) = D(x, p)U_0 D^\dagger(x, p) = D(2x, 2p)U_0 = U_0 D(-2x, -2p). \tag{5}$$

For later purposes we mention the following properties [3]:

$$U^\dagger(x, p) = U(x, p) \qquad U^2(x, p) = \mathbf{1} \tag{6}$$

$$U(x_1, p_1)U(x_2, p_2) = D(2x_1 - 2x_2, 2p_1 - 2p_2) \exp[2\mathrm{i}(x_1 p_2 - x_2 p_1)] \tag{7}$$

$$\mathrm{Tr}[D(x, p)] = 2\pi\delta(x)\delta(p). \tag{8}$$

The Wigner function of an 'arbitrary' operator Θ

$$\Theta = \sum_{N,M} \Theta_{NM} |N\rangle\langle M| \tag{9}$$

is defined in terms of the displaced parity operator as

$$\begin{aligned} W(x,p) &= \frac{1}{2\pi}\int \mathrm{d}X \left\langle x + \frac{1}{2}X \middle| \Theta \middle| x - \frac{1}{2}X \right\rangle \exp(-\mathrm{i}pX) \\ &= \frac{1}{\pi}\mathrm{Tr}[\Theta U(x,p)] = \sum_{N,M} \Theta_{NM} W_{MN}(x,p) \end{aligned} \tag{10}$$

where

$$\begin{aligned} W_{MN}(x,p) &\equiv \frac{1}{\pi}\langle M|U(x,p)|N\rangle \\ &= \frac{(-1)^N}{\pi}\left(\frac{N!}{M!}\right)^{1/2}[2^{1/2}(x+\mathrm{i}p)]^{M-N}\exp(-x^2-p^2)L_N^{M-N}(2x^2+2p^2). \end{aligned} \tag{11}$$

equations (3), (4) and (5) have been used in the proof of the above relation. Note that $W_{MN}(x,p)$ are complex functions. Combining equation (11) together with the property of equation (6) we easily prove that

$$W^*_{MN}(x,p) = W_{NM}(x,p). \tag{12}$$

This can also be proved using the expression in equation (11) in conjuction with the property of Laguerre polynomials

$$x^M (M!)^{-1} L_N^{M-N}(-x) = x^N (N!)^{-1} L_M^{N-M}(-x). \tag{13}$$

If Θ is a Hermitian operator $\Theta_{MN} = \Theta^*_{NM}$ we can easily prove that the corresponding Wigner function is real-valued. This is the case when Θ is a density matrix. For non-Hermitian operators the Wigner function is complex. For example if $\Theta = |f\rangle\langle g|$ where $|f\rangle$ and $|g\rangle$ are pure states, the corresponding 'cross-Wigner function' is, in general, complex. The study of such operators in this paper enables us to manipulate separately the cross-terms of a density matrix.

It has been pointed out in a more general context by Moyal [2], that $W_{MN}(x,p)$ form an orthonormal basis in the Hilbert space of complex functions of two variables. Indeed, we can prove the following properties [6]:

$$\sum_{M,N} W_{MN}(x,p) W^*_{MN}(x',p') = \frac{1}{2\pi}\delta(x-x')\delta(p-p') \tag{14}$$

$$\int \mathrm{d}x\,\mathrm{d}p\, W_{MN}(x,p) W^*_{M'N'}(x,p) = \frac{1}{2\pi}\delta_{MM'}\delta_{NN'}. \tag{15}$$

An easy way of proving equation (14) is to use equation (11) together with equations (7) and (8). Equation (15) can be proved by using equation (11) in conjuction with the properties of Laguerre polynomials [7]. From equations (15) and (10) we may express the density matrix elements Θ_{NM} in terms of the Wigner function as

$$\Theta_{NM} = 2\pi \int \mathrm{d}x\,\mathrm{d}p\, W(x,p) W^*_{MN}(x,p). \tag{16}$$

It is clear that to every (normalizable) complex function $W(x,p)$ corresponds an operator. Reference [8] studied necessary conditions that the Wigner function should obey so that the corresponding operator is a density matrix.

Another useful property that can be proved using equations (11) and (7) is

$$\sum_N W_{MN}(x,p)W_{NK}(x',p') = \frac{(-1)^K}{\pi}\exp\left[\mathrm{i}2(xp'-x'p)\right]W_{MK}(x-x',p-p'). \tag{17}$$

The Fourier transform of $W_{MN}(x,p)$ defined as

$$\tilde{W}_{MN}(X,P) = \int \mathrm{d}x\,\mathrm{d}p\; W_{MN}(x,p)\exp\left[-\mathrm{i}(Px-Xp)\right] \tag{18}$$

is the same function as $W_{MN}(x,p)$ up to a factor and with $x = X/2$ and $p = P/2$:

$$\begin{aligned}\tilde{W}_{MN}(X,P) &= \pi(-1)^N W_{MN}\left(\frac{X}{2},\frac{P}{2}\right)\\ &= \left(\frac{N!}{M!}\right)^{1/2}\left(\frac{X+\mathrm{i}P}{2^{1/2}}\right)^{M-N}\exp\left[-\frac{1}{4}(X^2+P^2)\right]L_N^{M-N}\left(\frac{X^2+P^2}{2}\right).\end{aligned} \tag{19}$$

Another function which is useful in phase space methods is the Weyl (or characteristic) function, which is defined as

$$\tilde{W}(X,P) = \int \mathrm{d}x\;\left\langle x+\tfrac{1}{2}X\,|\Theta|\,x-\tfrac{1}{2}X\right\rangle\exp(-\mathrm{i}Px) = \mathrm{Tr}\left[\Theta D(X,P)\right]. \tag{20}$$

We have used the notation $\tilde{W}(X,P)$ because it is known that the Wigner function is related to the Weyl function through the Fourier transform

$$\tilde{W}(X,P) = \int \mathrm{d}x\,\mathrm{d}p\;\; W(x,p)\exp\left[-\mathrm{i}(Px-Xp)\right] \tag{21}$$

with (x, P) and (X, p) as dual variables. Using the above formulas we can also express the Weyl function as

$$\tilde{W}(X,P) = \sum_{N,M}\Theta_{NM}\tilde{W}_{MN}(X,P) \tag{22}$$

where $\tilde{W}_{MN}(X,P)$ is defined in equations (18) and (19), and can also be written as

$$\tilde{W}_{MN}(X,P) = \langle M|D(X,P)|N\rangle. \tag{23}$$

We can express the elements Θ_{NM} in terms of the Weyl function as

$$\Theta_{NM} = \frac{1}{2\pi}\int \mathrm{d}X\,\mathrm{d}P\;\tilde{W}(X,P)\tilde{W}^*_{MN}(X,P). \tag{24}$$

We can also prove that

$$\frac{1}{2\pi}\int|\tilde{W}(X,P)|^2\,\mathrm{d}X\,\mathrm{d}P = 2\pi\int|W(x,p)|^2\,\mathrm{d}x\,\mathrm{d}p = \mathrm{Tr}[\Theta^\dagger\Theta]. \tag{25}$$

We note that the Weyl functions of the operators Θ and $\Theta^\dagger$ are related as follows:

$$\tilde{W}^*(-X,-P;\Theta) = \tilde{W}(X,P;\Theta^\dagger). \tag{26}$$

3. Correlation widths and uncertainties

Let Δx and Δp be the 'usual' uncertainties:

$$\begin{aligned}\langle x^M\rangle &= \int x^M\langle x|\Theta|x\rangle\,\mathrm{d}x = \int x^M W(x,p)\,\mathrm{d}x\,\mathrm{d}p\\ \Delta x &= (\langle x^2\rangle - \langle x\rangle^2)^{1/2}.\end{aligned} \tag{27}$$

Similar definitions hold for Δp. It is well known that if the operators Θ are density matrices,

$$\Delta x \Delta p \geqslant \tfrac{1}{2}. \tag{28}$$

We will see below examples of operators which are not density matrices and do not satisfy the inequality (28).

We next introduce the uncertainties

$$\begin{aligned} \langle\langle x^M \rangle\rangle &\equiv [\mathrm{Tr}(\Theta^\dagger \Theta)]^{-1} 2\pi \int x^M |W(x,p)|^2 \, \mathrm{d}x \, \mathrm{d}p \\ \sigma(x) &= (\langle\langle x^2 \rangle\rangle - \langle\langle x \rangle\rangle^2)^{1/2}. \end{aligned} \tag{29}$$

In a similar way we define for p. We note that the Wigner function in the first power appears in equation (27) and the squared Wigner function appears in equation (29). We also introduce the 'correlation width'

$$\begin{aligned} \langle\langle X^M \rangle\rangle &\equiv [\mathrm{Tr}(\Theta^\dagger \Theta)]^{-1} \frac{1}{2\pi} \int X^M |\tilde{W}(X,P)|^2 \, \mathrm{d}X \, \mathrm{d}P \\ \tau(X) &= (\langle\langle X^2 \rangle\rangle - \langle\langle X \rangle\rangle^2)^{1/2} \end{aligned} \tag{30}$$

and in a similar way we define the correlation width for P. Similar quantities have been introduced in [4] but here the normalization is different. The $\tau(X)$ and $\tau(P)$ quantify the correlations. The concept of quantum correlations is intimately connected with quantum mechanics. The formalism developed in this paper (X–P correlations plane, Weyl functions, $\tau(X)$, $\tau(P)$) tries to highlight and quantify the quantum correlations.

Using equation (26) we easily see that for Hermitian operators $\langle\langle X \rangle\rangle = \langle\langle P \rangle\rangle = 0$, but for more general operators considered here this is not necessarily true. For example, if $\Theta = |\alpha_1, \alpha_2\rangle\langle\beta_1, \beta_2|$ where $|\alpha_1, \alpha_2\rangle$ and $|\beta_1, \beta_2\rangle$ are coherent states, $\langle\langle X \rangle\rangle = \alpha_1 - \beta_1$, $\langle\langle P \rangle\rangle = \alpha_2 - \beta_2$. Such an operator is a cross-term in the density matrix describing the Schrödinger cat $\mathcal{N}(|\alpha_1, \alpha_2\rangle + |\beta_1, \beta_2\rangle)$. Here $\langle\langle X \rangle\rangle$, $\langle\langle P \rangle\rangle$ define the 'average position' of the corresponding cross-Weyl function in the X–P plane.

It can be proved [4] that for *pure states* ($\Theta = |s\rangle\langle s|$)

$$\sigma(x) = 2^{-1/2}\Delta x \quad \sigma(p) = 2^{-1/2}\Delta p \quad \tau(X) = 2^{1/2}\Delta x \quad \tau(P) = 2^{1/2}\Delta p \tag{31}$$

but in general the σ- and τ-quantities are different from the Δ-uncertainties. Reference [4] proved the inequalities

$$\tau(X)\sigma(p) \geqslant \tfrac{1}{2} \qquad \sigma(x)\tau(P) \geqslant \tfrac{1}{2} \tag{32}$$

written here in a form consistent with the normalization that we use in this paper.

As an example we consider the thermal density matrices

$$\begin{aligned} \rho_{\mathrm{th}} &= [1 - \exp(-\beta)] \sum_N \exp(-\beta N)|N\rangle\langle N| \\ &= \frac{1}{2\pi\eta_T} \int \mathrm{d}\gamma_1 \, \mathrm{d}\gamma_2 \exp\left(-\frac{\gamma_1^2 + \gamma_2^2}{2\eta_T}\right) |\gamma_1, \gamma_2\rangle\langle\gamma_1, \gamma_2| \end{aligned} \tag{33}$$

where $\eta_T = (\exp\beta - 1)^{-1}$ is the average number of thermal photons. It can easily be shown that the corresponding Wigner and Weyl functions are

$$W(x,p) = \frac{\mu}{\pi} \exp[-\mu(x^2 + p^2)] \tag{34}$$

$$\tilde{W}(X,P) = \exp\left[-\frac{1}{4\mu}(X^2 + P^2)\right] \tag{35}$$

where $\mu = \tanh(\beta/2) = (1+2\eta_T)^{-1}$. More generally we consider the above Gaussian Wigner and Weyl functions for any positive μ. Equation (16) shows that the corresponding operator is

$$\Theta_{NM} = 2\mu \frac{(1-\mu)^N}{(1+\mu)^{N+1}} \delta_{NM} \tag{36}$$

where δ_{NM} is the Kronecker delta. It is seen that when $0 < \mu < 1$ this is the thermal density matrix. For $\mu = 1$, it is simply the projection operator $|0\rangle\langle 0|$ to the vacuum. When $\mu > 1$, this is an operator with both positive and negative eigenvalues, which is not a density matrix.

Using the Wigner and Weyl functions of equations (34) and (35) we calculate the quantities

$$\Delta x = \Delta p = (2\mu)^{-1/2} \qquad \sigma(x) = \sigma(p) = \frac{1}{2\sqrt{\mu}} \qquad \tau(X) = \tau(P) = \sqrt{\mu}. \tag{37}$$

It is seen that for the Gaussian Wigner and Weyl functions of equations (34) and (35) the inequalities of equation (32) become equalities. Indeed, we can easily show that in this case $\mathrm{Tr}[\Theta^\dagger \Theta] = \mu$. It is also seen that for $\mu > 1$, where the operator Θ is *not* a density matrix, the uncertainty product $\Delta x \Delta p$ is less than $1/2$.

Intuitively somebody might expect that any measure of correlation (we use $\tau(X)$, $\tau(P)$) is related to the uncertainties in position and momentum (which we quantify not only with Δx, Δp but also with $\sigma(x)$, $\sigma(p)$). Equation (31) shows that this is true for *pure states*. We see in equation (37) that for mixed states, 'the mixture' increases the uncertainties and decreases the correlations.

As a second example we consider the mixed state

$$\rho = \tfrac{1}{2}(|\alpha_1, \alpha_2\rangle\langle\alpha_1, \alpha_2| + |-\alpha_1, -\alpha_2\rangle\langle-\alpha_1, -\alpha_2|). \tag{38}$$

In this case

$$\Delta x = \left(\tfrac{1}{2} + \alpha_1^2\right)^{1/2} \qquad \Delta p = \left(\tfrac{1}{2} + \alpha_2^2\right)^{1/2} \tag{39}$$

$$\sigma(x) = \frac{1}{2}\left[1 + \frac{4\alpha_1^2}{1 + \exp\left(-2\alpha_1^2 - 2\alpha_2^2\right)}\right]^{1/2} \qquad \sigma(p) = \frac{1}{2}\left[1 + \frac{4\alpha_2^2}{1 + \exp\left(-2\alpha_1^2 - 2\alpha_2^2\right)}\right]^{1/2} \tag{40}$$

$$\tau(X) = \left[1 - \frac{4\alpha_2^2}{1 + \exp\left(2\alpha_1^2 + 2\alpha_2^2\right)}\right]^{1/2} \qquad \tau(P) = \left[1 - \frac{4\alpha_1^2}{1 + \exp\left(2\alpha_1^2 + 2\alpha_2^2\right)}\right]^{1/2}. \tag{41}$$

These results should be compared and contrasted with those for the pure state

$$|s\rangle = \left[2 + 2\exp\left(-\alpha_1^2 - \alpha_2^2\right)\right]^{-1/2} (|\alpha_1, \alpha_2\rangle + |-\alpha_1, -\alpha_2\rangle) \tag{42}$$

which are

$$\begin{gathered}\Delta x = \left[\frac{1}{2} + \frac{\alpha_1^2 - \alpha_2^2 \exp\left(-\alpha_1^2 - \alpha_2^2\right)}{1 + \exp\left(-\alpha_1^2 - \alpha_2^2\right)}\right]^{1/2} \qquad \Delta p = \left[\frac{1}{2} + \frac{\alpha_2^2 - \alpha_1^2 \exp\left(-\alpha_1^2 - \alpha_2^2\right)}{1 + \exp\left(-\alpha_1^2 - \alpha_2^2\right)}\right]^{1/2} \\ \sigma(x) = 2^{-1/2}\Delta x \quad \sigma(p) = 2^{-1/2}\Delta p \quad \tau(X) = 2^{1/2}\Delta x \quad \tau(P) = 2^{1/2}\Delta p.\end{gathered} \tag{43}$$

It is seen that for large α_1 and α_2, in the case of the mixed state of equation (38) the correlation widths are small ($\tau(X) \approx \tau(P) \approx 1$); while in the case of the pure state of equation (42) the correlation widths are large ($\tau(X) \approx \alpha_1\sqrt{2}$ and $\tau(P) \approx \alpha_2\sqrt{2}$). It is also seen that for large α_1 and α_2, in the case of the mixed state of equation (38) the uncertainty widths are $\sigma(x) \approx \alpha_1$ and $\sigma(p) \approx \alpha_2$; while in the case of the pure state of equation (42) the uncertainty widths are $\sigma(x) \approx \alpha_1/\sqrt{2}$ and $\sigma(p) \approx \alpha_2/\sqrt{2}$, respectively.

4. Creation and annihilation operators for Wigner and Weyl functions

The Fourier transform between the Wigner and Weyl functions encourages us to develop a 'harmonic oscillator formalism' with the Wigner *x–p* and Weyl *X–P* as dual representations. In the Wigner *x–p* representation

$$\hat{x} = x \qquad \hat{p} = p \qquad \hat{X} = \mathrm{i}\partial_p \qquad \hat{P} = -\mathrm{i}\partial_x \tag{44}$$

$$[\hat{x}, \hat{p}] = [\hat{x}, \hat{X}] = [\hat{p}, \hat{P}] = [\hat{X}, \hat{P}] = 0 \qquad [\hat{x}, \hat{P}] = [\hat{X}, \hat{p}] = \mathrm{i}. \tag{45}$$

We introduce the following ladder operators

$$\begin{aligned} b_1 &= 2^{-1/2}(\hat{x} - \mathrm{i}\hat{p}) - 2^{-3/2}(\hat{X} - \mathrm{i}\hat{P}) \qquad b_1^\dagger = 2^{-1/2}(\hat{x} + \mathrm{i}\hat{p}) - 2^{-3/2}(\hat{X} + \mathrm{i}\hat{P}) \\ b_2 &= 2^{-1/2}(\hat{x} + \mathrm{i}\hat{p}) + 2^{-3/2}(\hat{X} + \mathrm{i}\hat{P}) \qquad b_2^\dagger = 2^{-1/2}(\hat{x} - \mathrm{i}\hat{p}) + 2^{-3/2}(\hat{X} - \mathrm{i}\hat{P}). \end{aligned} \tag{46}$$

It is easily seen that

$$[b_1, b_2] = [b_1, b_2^\dagger] = [b_1^\dagger, b_2] = [b_1^\dagger, b_2^\dagger] = 0 \qquad [b_1, b_1^\dagger] = [b_2, b_2^\dagger] = \mathbf{1}. \tag{47}$$

Straightforward, albeit lengthy, differentiation shows that when these operators act upon $W_{MN}(x, p)$, they give

$$\begin{aligned} b_1 W_{MN}(x, p) &= M^{1/2} W_{M-1,N}(x, p) \\ b_1^\dagger W_{MN}(x, p) &= (M+1)^{1/2} W_{M+1,N}(x, p) \\ b_2 W_{MN}(x, p) &= N^{1/2} W_{M,N-1}(x, p) \\ b_2^\dagger W_{MN}(x, p) &= (N+1)^{1/2} W_{M,N+1}(x, p). \end{aligned} \tag{48}$$

It is seen that $b_1^\dagger, b_2^\dagger$ and b_1, b_2 are creation and annihilation operators for Wigner functions. W_{MN} are eigenstates of the 'number operators' $b_1^\dagger b$ and $b_2^\dagger b_2$ with eigenvalues *M* and *N*, respectively:

$$b_1^\dagger b_1 W_{MN}(x, p) = M W_{MN}(x, p) \qquad b_2^\dagger b_2 W_{MN}(x, p) = N W_{MN}(x, p). \tag{49}$$

We also point out that

$$\begin{aligned} b_1 \langle M|U(x, p)|N\rangle &= \langle M|a^\dagger U(x, p)|N\rangle \qquad b_1^\dagger \langle M|U(x, p)|N\rangle = \langle M|aU(x, p)|N\rangle \\ b_2 \langle M|U(x, p)|N\rangle &= \langle M|U(x, p)a|N\rangle \qquad b_2^\dagger \langle M|U(x, p)|N\rangle = \langle M|U(x, p)a^\dagger|N\rangle. \end{aligned} \tag{50}$$

More generally, for arbitrary functions $f(a, a^\dagger)$ and $g(a, a^\dagger)$ we have

$$f(b_1^\dagger, b_1) g(b_2, b_2^\dagger) \langle M|U(x, p)|N\rangle = \langle M|f(a, a^\dagger)U(x, p)g(a, a^\dagger)|N\rangle. \tag{51}$$

Using this equation we prove that

$$W(x, p; f(a, a^\dagger)\Theta g(a, a^\dagger)) = g(b_1^\dagger, b_1) f(b_2, b_2^\dagger) W(x, p). \tag{52}$$

The above operators can also be written in the Weyl *X–P* representation with

$$\hat{x} = \mathrm{i}\partial_P \qquad \hat{p} = -\mathrm{i}\partial_X \qquad \hat{X} = X \qquad \hat{P} = P. \tag{53}$$

In this form they can act on Weyl functions

$$\begin{aligned} b_1 \tilde{W}_{MN}(X, P) &= -M^{1/2} \tilde{W}_{M-1,N}(X, P) \\ b_1^\dagger \tilde{W}_{MN}(X, P) &= -(M+1)^{1/2} \tilde{W}_{M+1,N}(X, P) \\ b_2 \tilde{W}_{MN}(X, P) &= -N^{1/2} \tilde{W}_{M,N-1}(X, P) \\ b_2^\dagger \tilde{W}_{MN}(X, P) &= -(N+1)^{1/2} \tilde{W}_{M,N+1}(X, P) \end{aligned} \tag{54}$$

and

$$b_1^\dagger b_1 \tilde{W}_{MN}(X, P) = M \tilde{W}_{MN}(X, P) \qquad b_2^\dagger b_2 \tilde{W}_{MN}(X, P) = N \tilde{W}_{MN}(X, P). \tag{55}$$

5. Displacement of Wigner and Weyl functions

We define the displacement operators as

$$\mathcal{D}(\alpha_1, \alpha_2, \beta_1, \beta_2) \equiv \exp[\mathrm{i}(\alpha_2 - \beta_2)\hat{x}] \exp\left(-\mathrm{i}\frac{\alpha_1 + \beta_1}{2}\hat{P}\right) \times \exp(\mathrm{i}(\beta_1 - \alpha_1)\hat{p}) \exp\left(\mathrm{i}\frac{\alpha_2 + \beta_2}{2}\hat{X}\right). \quad (56)$$

The above notation might seem complicated here, but it is chosen because it simplifies a lot of formulas later. In the Wigner *x–p* representation we use equation (44) and find

$$\mathcal{D}(\alpha_1, \alpha_2, \beta_1, \beta_2) W(x, p) = \exp[\mathrm{i}(\alpha_2 - \beta_2)x] \exp(\mathrm{i}(\beta_1 - \alpha_1)p) \times W\left(x - \frac{\alpha_1 + \beta_1}{2}, p - \frac{\alpha_2 + \beta_2}{2}\right). \quad (57)$$

We note that if $W(x, p)$ corresponds to a Hermitian operator (e.g. a density matrix) and is therefore real, the $\mathcal{D}(\alpha_1, \alpha_2, \beta_1, \beta_2)W(x, p)$ is in general complex and the corresponding operator is no longer Hermitian.

In the Weyl *X–P* representation we use equation (53) and find

$$\mathcal{D}(\alpha_1, \alpha_2, \beta_1, \beta_2)\tilde{W}(X, P) = \exp\left(-\mathrm{i}\frac{\alpha_1 + \beta_1}{2}P + \mathrm{i}\frac{\alpha_2 + \beta_2}{2}X - \mathrm{i}\alpha_1\beta_2 + \mathrm{i}\alpha_2\beta_1\right) \times \tilde{W}(X + \beta_1 - \alpha_1, P - \alpha_2 + \beta_2). \quad (58)$$

Let Θ be the operator corresponding to the Wigner function $W(x, p)$ and Weyl function $\tilde{W}(X, P)$. Using equation (52) we can show that the operator $D(\alpha_1, \alpha_2)\Theta D^\dagger(\beta_1, \beta_2)$ corresponds to the Wigner fuction

$$W(x, p; D(\alpha_1, \alpha_2)\Theta D^\dagger(\beta_1, \beta_2)) = \exp\left[\frac{\mathrm{i}}{2}(\alpha_2\beta_1 - \alpha_1\beta_2)\right]\mathcal{D}(\alpha_1, \alpha_2, \beta_1, \beta_2)W(x, p). \quad (59)$$

This shows that when $\mathcal{D}(\alpha_1, \alpha_2, \beta_1, \beta_2)$ acts on a Wigner function, the corresponding operator Θ is displaced on the left and right with different (in general) displacements. In the special case [9] $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$ we get

$$W(x, p; D(\alpha_1, \alpha_2)\Theta D^\dagger(\alpha_1, \alpha_2)) = \mathcal{D}(\alpha_1, \alpha_2, \alpha_1, \alpha_2)W(x, p) = W(x - \alpha_1, p - \alpha_2). \quad (60)$$

6. Gaussian Wigner and Weyl functions

We displace the Gaussian Wigner function of equation (34), and get

$$G_\mu(x, p; \alpha_1, \alpha_2, \beta_1, \beta_2) \equiv \mathcal{D}(\alpha_1, \alpha_2, \beta_1, \beta_2)\frac{\mu}{\pi}\exp[-\mu(x^2 + p^2)]$$
$$= \frac{\mu}{\pi}\exp\left[-\mu\left(x - \frac{\alpha_1 + \beta_1}{2}\right)^2 + \mathrm{i}(\alpha_2 - \beta_2)x\right] \times \exp\left[-\mu\left(p - \frac{\alpha_2 + \beta_2}{2}\right)^2 + \mathrm{i}(\beta_1 - \alpha_1)p\right]. \quad (61)$$

Using equation (59) we interpret this Wigner function, for $0 < \mu \leqslant 1$, as representing the 'displaced thermal density matrix', which is displaced on the left and right with different (in general) displacements:

$$\rho(\alpha_1, \alpha_2, \beta_1, \beta_2) = D(\alpha_1, \alpha_2)\rho_{\mathrm{th}}D^\dagger(\beta_1, \beta_2)$$
$$= \frac{1}{2\pi\eta_T}\int \mathrm{d}\gamma_1\,\mathrm{d}\gamma_2 \exp\left(-\frac{\gamma_1^2 + \gamma_2^2}{2\eta_T}\right)\exp\left\{\frac{\mathrm{i}}{2}[\gamma_1(\alpha_2 - \beta_2) - \gamma_2(\alpha_1 - \beta_1)]\right\} \times |\gamma_1 + \alpha_1, \gamma_2 + \alpha_2\rangle\langle\gamma_1 + \beta_1, \gamma_2 + \beta_2|. \quad (62)$$

Here $\eta_T = (1-\mu)/(2\mu)$. When $\mu = 1$ it represents the operator $|\alpha_1, \alpha_2\rangle\langle\beta_1, \beta_2|$. When $\mu > 1$ it represents (as explained above) an operator which is not a density matrix, displaced on the left and right with different (in general) displacements.

We also introduce the displaced Gaussian Weyl functions as the Fourier transform of the displaced Gaussian Wigner functions

$$\begin{aligned}\tilde{G}_\mu(X, P; \alpha_1, \alpha_2, \beta_1, \beta_2) &\equiv \mathcal{D}(\alpha_1, \alpha_2, \beta_1, \beta_2) \exp\left(-\frac{X^2}{4\mu} - \frac{P^2}{4\mu}\right)\\ &= \exp\left[-\frac{(X+\beta_1-\alpha_1)^2}{4\mu} + \mathrm{i}\frac{\alpha_2+\beta_2}{2}(X+\beta_1-\alpha_1)\right]\\ &\quad\times \exp\left[-\frac{(P-\alpha_2+\beta_2)^2}{4\mu} - \mathrm{i}\frac{\alpha_1+\beta_1}{2}(P-\alpha_2+\beta_2)\right]. \end{aligned} \tag{63}$$

It is easily seen that for the G_μ and $\tilde{G}_\mu$

$$\langle\langle x\rangle\rangle = \frac{\alpha_1+\beta_1}{2} \qquad \langle\langle p\rangle\rangle = \frac{\alpha_2+\beta_2}{2} \tag{64}$$

$$\langle\langle X\rangle\rangle = \alpha_1 - \beta_1 \qquad \langle\langle P\rangle\rangle = \alpha_2 - \beta_2 \tag{65}$$

and that the 'σ-quantities' are the same as in equation (37).

In the case $\mu = 1$ it is easily seen that the Gaussian Wigner and Weyl functions are eigenstates of the annihilation operators

$$\begin{aligned} b_1 G_1(x, p; \alpha_1, \alpha_2, \beta_1, \beta_2) &= 2^{-1/2}(\beta_1 - \mathrm{i}\beta_2) G_1(x, p; \alpha_1, \alpha_2, \beta_1, \beta_2)\\ b_2 G_1(x, p; \alpha_1, \alpha_2, \beta_1, \beta_2) &= 2^{-1/2}(\alpha_1 + \mathrm{i}\alpha_2) G_1(x, p; \alpha_1, \alpha_2, \beta_1, \beta_2). \end{aligned} \tag{66}$$

Furthermore, $G_\mu(x, p; \alpha_1, \alpha_2, \beta_1, \beta_2)$ satisfies the 'resolution of the identity' relation

$$\begin{aligned}\frac{1}{2\pi\mu}\int &\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\mathrm{d}\beta_1\,d\beta_2\, G_\mu(x, p; \alpha_1, \alpha_2, \beta_1, \beta_2) G_\mu^*(x', p'; \alpha_1, \alpha_2, \beta_1, \beta_2)\\ &= \delta(x-x')\delta(p-p'). \end{aligned} \tag{67}$$

Using this we can expand an arbitrary Wigner function $W(x, p)$ as

$$W(x, p) = \frac{1}{2\pi\mu}\int \mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\mathrm{d}\beta_1\,\mathrm{d}\beta_2\,\lambda(\alpha_1, \alpha_2, \beta_1, \beta_2) G_\mu(x, p; \alpha_1, \alpha_2, \beta_1, \beta_2) \tag{68}$$

$$\lambda(\alpha_1, \alpha_2, \beta_1, \beta_2) = \int \mathrm{d}x\,\mathrm{d}p\, W(x, p) G_\mu^*(x, p; \alpha_1, \alpha_2, \beta_1, \beta_2). \tag{69}$$

When $\mu = 1$, this is, in the Wigner language, the expansion of an operator in terms of the $|\alpha_1, \alpha_2\rangle\langle\beta_1, \beta_2|$. When $0 < \mu < 1$, it is the expansion of an operator in terms of the displaced thermal density matrices of equation (62) and when $\mu > 1$ it is in terms of operators that are not density matrices.

Similar expansion can be made for an arbitrary Weyl function in terms of the Gaussian Weyl functions $\tilde{G}_\mu(X, P; \alpha_1, \alpha_2, \beta_1, \beta_2)$.

7. Squeezing of Wigner and Weyl functions

Squeezing in the x–p–X–P extended phase space will enable us to build states with desirable correlations $\tau(X)$ and $\tau(P)$. For example, for the Gaussian Wigner and Weyl functions of equations (34) and (35) which correspond to density matrices for $\mu \leqslant 1$, it can be seen from equation (37) that the maximum values of the correlations $\tau(X)$ and $\tau(P)$ are equal to 1. Squeezing transformations will lead to states with large correlations $\tau(X)$ and $\tau(P)$ and small $\sigma(p)$ and $\sigma(x)$.

Since from a mathematical point of view we have a two-mode formalism (which, however, is used for the description of Wigner and Weyl functions of one-mode systems), the general squeezing transformations (e.g. [11]) involve the $Sp(4, R)$ group with the ten generators $\hat{x}^2$, $\hat{p}^2$, $\hat{X}^2$, $\hat{P}^2$, $\hat{x}\hat{p}$, $\hat{x}\hat{X}$, $\hat{x}\hat{P}$, $\hat{p}\hat{X}$, $\hat{p}\hat{P}$ and $\hat{X}\hat{P}$. In this paper we consider the very special case where

$$S(r) = \exp[-\mathrm{i}r(\hat{x}\hat{P} + \hat{p}\hat{X})]. \tag{70}$$

Using the formula [12]

$$\exp[rx\partial_x]f(x) = f(x\mathrm{e}^r) \tag{71}$$

we act with the operator $S(r)$ (in the representation of equation (44)) on the Gaussian Wigner function of equation (34), and get

$$\exp[-rx\partial_x + rp\partial_p]\frac{\mu}{\pi}\exp[-\mu(x^2+p^2)] = \frac{\mu}{\pi}\exp[-\mu(\mathrm{e}^{-2r}x^2 + \mathrm{e}^{2r}p^2)]. \tag{72}$$

Similarly acting with the operator $S(r)$ (in the representation of equation (53)) on the Gaussian Weyl function of equation (35), we get

$$\exp[r\partial_P P - r\partial_X X]\exp\left[-\frac{1}{4\mu}(X^2+P^2)\right] = \exp\left[-\frac{1}{4\mu}(\mathrm{e}^{-2r}X^2 + \mathrm{e}^{2r}P^2)\right]. \tag{73}$$

For these Wigner and Weyl functions we calculate the quantities

$$\sigma(x) = \frac{\mathrm{e}^r}{2\sqrt{\mu}} \qquad \sigma(p) = \frac{\mathrm{e}^{-r}}{2\sqrt{\mu}} \qquad \tau(X) = \mathrm{e}^r\sqrt{\mu} \qquad \tau(P) = \mathrm{e}^{-r}\sqrt{\mu}. \tag{74}$$

It is seen that these states have large correlation in position ($\tau(X)$) and small uncertainty in momentum ($\sigma(p)$). We stress that the well-known concept of squeezing is seen here in a novel context. It is applied in the extended phase space *x–p–X–P* which shows not only the position and momentum but also correlations in position and momentum.

As already mentioned, the general case of squeezing operators in this space contains ten generators and requires further study which will reveal deeper connections between correlations and uncertainties and enable the researcher to construct (at least theoretically) states with desirable correlations and uncertainties, subject to the constraints of the uncertainty relations of equation (32). States with desirable values of $\tau(X)$ and $\tau(P)$ can be constructed with appropriate squeezing. In this paper we have provided the necessary tools for this. We have introduced creation and annihilation operators for the Wigner and Weyl functions $b_1, b_1^\dagger, b_2, b_2^\dagger$ (to be distinguished from the usual creation and annihilation operators $a, a^\dagger$), the corresponding displacement operators, etc.

8. Time evolution of Wigner and Weyl functions

Let $H(a, a^\dagger)$ be the Hamiltonian of the system and $\rho(0)$ its density matrix at time $t = 0$. At time t the density matrix of the system is

$$\rho(t) = \exp(\mathrm{i}tH)\rho(0)\exp(-\mathrm{i}tH). \tag{75}$$

We now introduce

$$G(b_1, b_1^\dagger, b_2, b_2^\dagger) \equiv H(b_2, b_2^\dagger) - H(b_1^\dagger, b_1) \tag{76}$$

and using equation (52) we write the Wigner function of the system at time t as

$$W(x, p; \rho(t)) = \exp[\mathrm{i}tG(b_1, b_1^\dagger, b_2, b_2^\dagger)]W(x, p; \rho(0)). \tag{77}$$

In this sense $G(b_1, b_1^\dagger, b_2, b_2^\dagger)$ is the 'Hamiltonian' for the Wigner function. This looks like a Hamiltonian of a two-mode system but is used for the description of an one-mode system. We note that a two-mode Hamiltonian for the description of an one-mode system is also used in thermo-field dynamics [10].

In the presence of dissipation the evolution of the density matrix is described with the equation

$$\partial_t \rho = \mathrm{i}[H, \rho] + \frac{\gamma}{2}(M+1)(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a) + \frac{\gamma}{2}M(2a^\dagger \rho a - aa^\dagger \rho - \rho aa^\dagger) \tag{78}$$

where γ is the damping rate and M is the average number of bath quanta. For a monochromatic bath with frequency ω_B in thermal equilibrium at temperature T, $M(\omega_B) = [\exp(\beta\omega_B) - 1]^{-1}$. Using equation (51) we easily see that the Wigner function evolves according to the equation

$$\begin{aligned}\partial_t W(x, p) = \mathrm{i}GW(x, p) &+ \frac{\gamma}{2}(M+1)(2b_2 b_1 - b_2^\dagger b_2 - b_1 b_1^\dagger)W(x, p)\\ &+ \frac{\gamma}{2}M(2b_2^\dagger b_1^\dagger - b_2 b_2^\dagger - b_1^\dagger b_1)W(x, p).\end{aligned} \tag{79}$$

So the operator equation (78) which involves non-commuting operators reduces conveniently to the differential equation (79).

Similar equations can be written for the Weyl function.

9. Discussion

The Wigner–Weyl formalism provides a deeper insight into the properties of a quantum state that cannot be easily seen from the wavefunction (either in x or in p-representation). The Wigner function shows that the position and momentum of the particle in a way are consistent with the uncertainty principle. The Weyl function shows quantum correlations. In this paper we have developed tools for the design and construction of quantum states at a finer level, namely in the x–p–X–P space using the Wigner x–p and Weyl X–P representations. This formalism is also suitable for the study of general operators (which are not necessarily density matrices).

The correlation properties of a state have been quantified with the correlation widths $\tau(X)$ and $\tau(P)$. The values of these quantities have been calculated for several examples. The correlation widths $\tau(X)$ and $\tau(P)$ in conjuction with the uncertainties $\sigma(x)$ and $\sigma(p)$, satisfy the inequalities of equation (32).

The 'building blocks' for the Wigner and Weyl functions are the $W_{MN}(x, p)$ of equation (11) which involve Laguerre polynomials. We have introduced ladder operators for these functions which are different from the usual creation and annihilation operators (which act on number states). We have also introduced displacement operators for the Wigner and Weyl functions of an operator Θ. They are functions of four variables and their physical meaning has been explained in equation (59) which shows that the corresponding operator Θ is displaced on the left and right with different (in general) displacements. Gaussian Wigner and Weyl functions and their displacements, have been studied in section 6.

Squeezing of these functions has been studied in section 7. It has been shown that appropriate squeezing can lead to states with large correlations (equation (74)). We stress that this is squeezing in the x–p–X–P extended phase space and its general case contains ten generators. It is therefore much more general than the well-known squeezing in position and momentum and enables the researcher to manipulate the correlation properties of quantum states.

In section 8 the formalism has been used for the study of the evolution of quantum systems (with or without dissipation). It can also be used in the context of quantum tomography (e.g. [13]) or Bose–Einstein condensates [14].

References

[1] Balazs N L and Jennings B K 1984 *Phys. Rep.* **104** 347
Hillery M, O'Connell R F, Scully M O and Wigner E P 1984 *Phys. Rep.* **106** 121
Lee H W 1995 *Phys. Rep.* **259** 147
[2] Moyal J E 1949 *Proc. Camb. Phil. Soc.* **45** 99
Bartlett M S and Moyal J E 1949 *Proc. Camb. Phil. Soc.* **45** 545
Berezin F A 1974 *Math. USSR Izv.* **8** 1109
Berezin F A 1975 *Math. USSR Izv.* **9** 341
Berezin F A 1975 *Commun. Math. Phys.* **40** 153
[3] Grossmann A 1976 *Commun. Math. Phys.* **48** 191
Englert B G 1989 *J. Phys. A: Math. Gen.* **22** 625
Royer A 1991 *Phys. Rev.* A **43** 44
Bishop R F and Vourdas A 1994 *Phys. Rev.* A **50** 4488
Chountasis S and Vourdas A 1998 *Phys. Rev.* A **58** 848
[4] Chountasis S and Vourdas A 1998 *Phys. Rev.* A **58** 1794
Chountasis S and Vourdas A 1999 *J. Phys. A: Math. Gen.* **32** 6949
[5] Pomarenko S A and Wolf E 2001 *Phys. Rev.* A **63** 062106
[6] Wuensche A 1998 *J. Phys. A: Math. Gen.* **31** 8267
Wuensche A 1999 *J. Phys. A: Math. Gen.* **32** 3179
[7] Gradshteyn I S and Ryzhik I M 1994 *Table of Integrals Series and Products* (London: Academic) p 844
[8] Narcowich F J and O'Connell R F 1986 *Phys. Rev.* A **34** 1
[9] Zak J 1999 *J. Phys. A: Math. Gen.* **32** 4787
[10] Umezawa H, Matsumoto H and Tachiki M 1987 *Thermofield Dynamics and Condensed States* (Amsterdam: North-Holland)
[11] Bishop R F and Vourdas A 1998 *Z. Phys.* B **71** 527
Kim Y S and Noz M E 1991 *Phase Space Picture of Quantum Mechanics* (Singapore: World Scientific)
[12] Nieto M M 1996 *Quantum Semiclass. Opt.* **8** 1061
Nieto M M 1997 *Phys. Lett.* A **229** 135
[13] Vogel K and Risken H 1989 *Phys. Rev.* A **40** 2847
Smithey D T, Beck M, Raymer M G and Faridani T 1993 *Phys. Rev. Lett.* **70** 1244
Leibfried D *et al* 1996 *Phys. Rev. Lett.* **77** 4281
Kurtsiefer C, Pfau T and Mlynek J 1997 *Nature* **386** 150
Breitenbach G, Schiller S and Mlynek J 1997 *Nature* **387** 471
[14] Franke-Arnold S, Huyet G and Barnett S M 2001 *J. Phys. B: At. Mol. Opt. Phys.* **34** 945